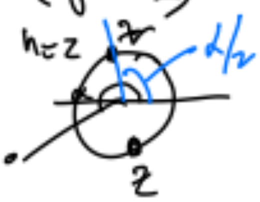
Solving complex # equations.

One type of example:

$$z^n = c$$ ← some complex #.

In this case, there are n distinct solutions, and they are the vertices of a regular n-gon centered at $0 \in \mathbb{C}$.

(if $n > 2$)



To solve: Let $z = re^{i\theta}$, $c = Re^{i\alpha}$

$$\Rightarrow \quad r^n e^{in\theta} = c = Re^{i\alpha}$$

$$\Rightarrow r^n = R \Rightarrow r = R^{1/n} \in (0, \infty).$$

From $e^{in\theta} = e^{i\alpha} \Leftrightarrow n\theta = \alpha + 2k\pi$ for some $k \in \mathbb{Z}$.

$$\Rightarrow \theta = \frac{\alpha}{n} + \frac{2k\pi}{n} \text{ for } k \in \mathbb{Z}.$$

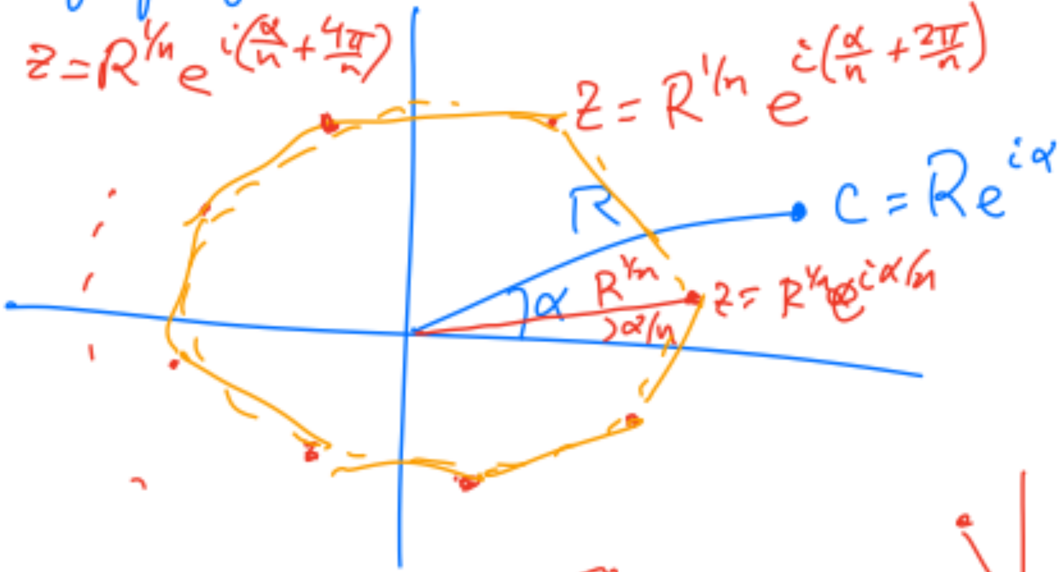
$$\Rightarrow \theta = \frac{\alpha}{n} + \frac{2k\pi}{n} \text{ for } k \in \{0, 1, 2, \dots, n-1\}$$

Since if $k = 3n$, we get the same answer for $e^{i\theta}$.

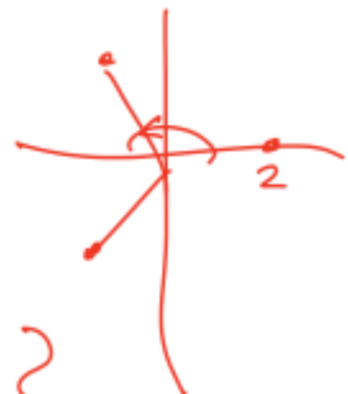
$\Rightarrow$ Solutions:

$$z = R^{1/n} e^{i\left(\frac{\alpha}{n} + \frac{2k\pi}{n}\right)} \quad \text{with } k \in \{0, 1, 2, \dots, n-1\}$$

graph of solutions to $z^n = Re^{i\alpha} = c$



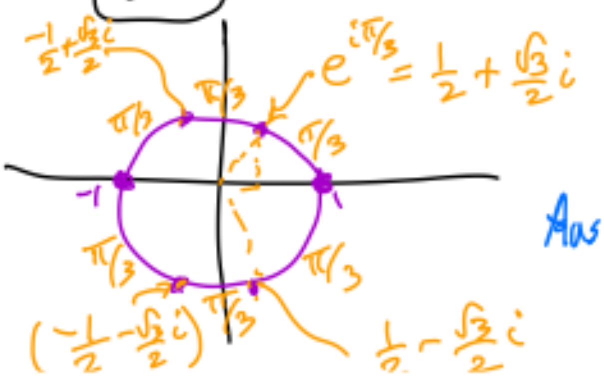
eg $z^3 = 8 \Rightarrow \left\{ z = 2 \text{ or } z = 2e^{i2\pi/3} \text{ or } z = 2e^{i4\pi/3} \right\}$



i.e. $z = 2$ or $2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ or $2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$\Rightarrow \boxed{z = 2 \text{ or } -1+\sqrt{3}i \text{ or } -1-\sqrt{3}i}$

$\boxed{\text{Ex}}$ Solve $x^6 = 1$ (answer: the 6^{th} roots of unity)



Ans: $x = 1, -1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i,$
$-\frac{1}{2} + \frac{\sqrt{3}}{2}i,$ or $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Ex Solve $w^3 + 4\sqrt{2} + 4i\sqrt{2} = 0$

$$w^3 = -4\sqrt{2} + -(4\sqrt{2})i = 4\sqrt{2}\underbrace{(-1+-i)}_{\text{length } \sqrt{2}}$$

$$w^3 = Re^{i\alpha} \qquad R = 4\sqrt{2}\cdot\sqrt{2} = 8$$

$$\alpha = \frac{5\pi}{4}.$$

$$\boxed{w^3 = 8e^{i\left(\frac{5\pi}{4}\right)}}$$

$w = re^{i\theta}$ $\quad$ $r^3 = 8$ & $3\theta = \frac{5\pi}{4} + 2k\pi$ for $k \in \mathbb{Z}$.

one root

$$w = 2e^{i\left(\frac{5\pi}{12}\right)}$$

other angles $\frac{5\pi}{12} + \frac{2\pi}{3}, \ \frac{5\pi}{12} + \frac{4\pi}{3}$

$$= \underbrace{\frac{(5+8)\pi}{12}}_{\frac{13\pi}{12}}, \ \underbrace{\frac{5\pi}{12} + \frac{16\pi}{12}}_{\frac{21\pi}{12}} =$$

Answers $\boxed{w = 2e^{i5\pi/12}, \text{ or } 2e^{i\frac{13\pi}{12}}, \text{ or } 2e^{i\frac{21\pi}{12}}}$

Example Solve for z:

$$(z+2)^4 = z^4$$

Q: how many solutions should I expect? (after writing $f(z) = 0$, $f(z)$ is a cubic poly: 3 solutions.)

We want to divide by z^4 | Note: If $z=0$, $(0+2)^4=0$ false $\Rightarrow z\neq 0$. ✓

$$\Rightarrow \frac{(z+2)^4}{z^4}=1 \Rightarrow \left(\frac{z+2}{z}\right)^4=1$$

$$\Rightarrow \left(1+\frac{2}{z}\right)^4=1$$

4th root root of 1

$$\Rightarrow 1+\frac{2}{z}=1,\ i,\ -1,\ -i.$$

$$\Rightarrow 1+\frac{2}{z}=1$$ No solution or $1+\frac{2}{z}=-1$

or $1+\frac{2}{z}=i$ or $1+\frac{2}{z}=-i$

$\Rightarrow$ possible answers:

$$\frac{2}{z}=i-1$$

$$\frac{z}{2}=\frac{1}{i-1} \Rightarrow z=\frac{2(-i-1)}{(i-1)(-i-1)}=\frac{2(-i-1)}{2}=\boxed{-i-1}.$$

$$\frac{2}{z}=-2 \Rightarrow \boxed{z=-1}$$

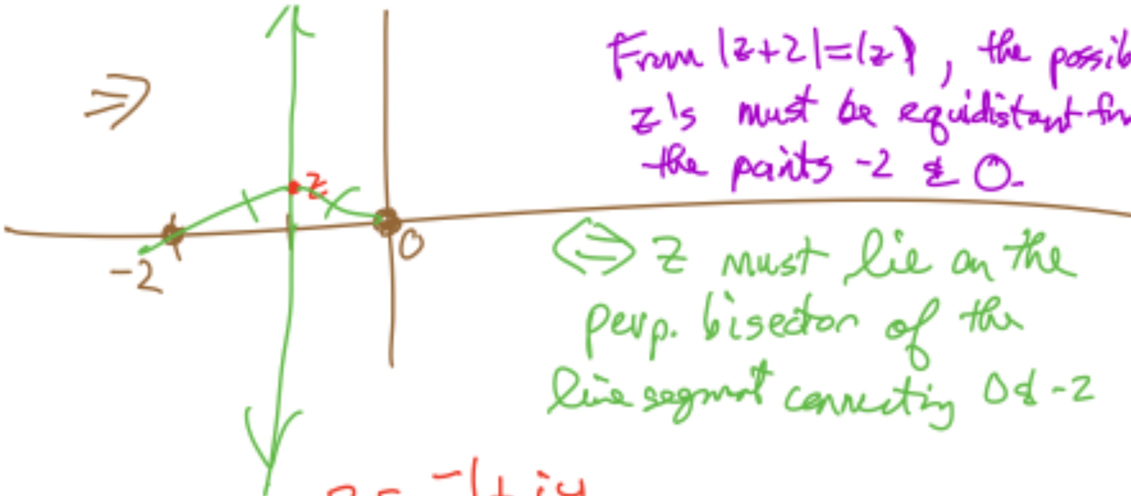
$$\frac{2}{z}=-i-1 \Rightarrow \frac{z}{2}=\frac{1}{-i-1} \Rightarrow z=\frac{2(i-1)}{\cancel{(-i-1)(+i-1)}\ 2}$$

$$\Rightarrow \boxed{z=i-1}.$$

Solutions: $\boxed{z=-1,\ -i-1,\ \text{or}\ -i-1}$

Solution #2: $(z+2)^4=z^4$ dist from z to -2 dist from z to 0

$$\Rightarrow |z+2|^4=|z|^4 \Rightarrow |z+2|=|z|$$



From $|z+2| = |z|$, the possible z's must be equidistant from the points -2 & 0.

$\Leftrightarrow$ z must lie on the perp. bisector of the line segment connecting 0 & -2

$$z = -1 + iy.$$

$$\Rightarrow (-1+iy+2)^4 = (-1+iy)^4$$

$$\Rightarrow (1+iy)^4 = (-1+iy)^4$$

If we write $w = 1+iy = re^{i\theta}$, $(-1+iy)^4 = (1-iy)^4 = \bar{\omega}^4$.

$$r^4 e^{i4\theta} = r^4 e^{-i4\theta}$$

$$\Rightarrow 4\theta = -4\theta + 2k\pi \text{ for } k \in \mathbb{Z}.$$

$$\Rightarrow 8\theta = 2k\pi \text{ for } k \in \mathbb{Z}.$$

$$\theta = \frac{k\pi}{4} \text{ for } k \in \mathbb{Z}.$$

The only solutions $1+iy = re^{i\theta}$ are when $k = 1, -1$, $k=0$

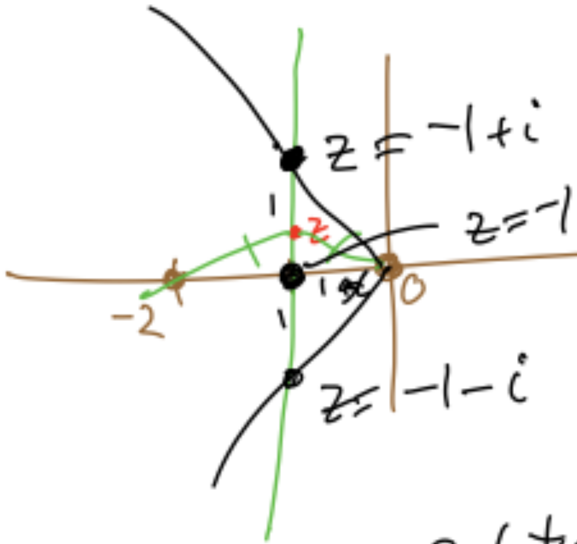
$$\overline{\left(e^{i\theta}\right)} = e^{-i\theta}$$

$$\omega = re^{i\pi/4},\ re^{-i\pi/4},\ r$$

$$z = -1+iy = -\bar{\omega} = \bar{\omega} e^{i\pi}$$

$$z = re^{-i\pi/4 + i\pi},\ re^{\frac{i\pi}{4} + i\pi}$$

$$z = re^{i\frac{3\pi}{4}},\ re^{i\frac{5\pi}{4}},\ re^{i\pi}$$



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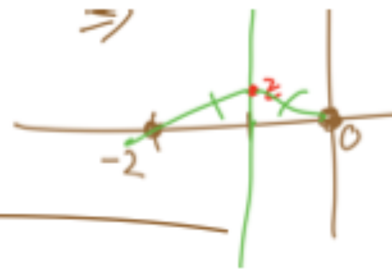


∴ Solution are $\boxed{z = -1, -1+i, -1-i}$

Solution #3 $\quad (z+2)^4 = z^4$

Pascal's Δ $\Rightarrow z^4 + 4z^3\cdot 2 + 6z^2\cdot 2^2 + 4\cdot z\, 2^3 + 2^4 = z^4$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$\Rightarrow 8z^3 + 24z^2 + 32z + 16 = 0$

$\Rightarrow z^3 + 3z^2 + 4z + 2 = 0$

Rational Roots Thm $\quad z = \frac{\pm \text{factors of } 2}{\text{factors of } 1} \leadsto z = -1$ solves it.

$$
\begin{array}{r|l}
 & z^2 + 2z + 2 \\
\hline
z+1 & z^3 + 3z^2 + 4z + 2 \\
 & \underline{z^3 + z^2} \\
 & 2z^2 + 4z + 2 \\
 & \underline{2z^2 + 2z} \\
 & 2z + 2 \\
 & \underline{2z + 2} \\
 & 0
\end{array}
$$

$\Rightarrow z^3 + 3z^2 + 4z + 2 = (z+1)(z^2 + 2z + 2) = 0$

$$\Rightarrow z = -1 \text{ or } z = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$\Rightarrow z = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$\Rightarrow z = -1 \pm i$$

$$\Rightarrow \boxed{z = -1 \text{ or } -1+i \text{ or } -1-i}$$

New topic: Set Theory & Topology

• Defns and Theorems to memorize that we will use in the class.

• $B(z, \varepsilon)$ z fixed complex #, $\varepsilon > 0$, i.e. ε is a + real #.

= open Disk of radius ε centered at z.
"ball"

$$= \{ w \in \mathbb{C} : |z - w| < \varepsilon \}$$

• $\overline{B(z, \varepsilon)}$ = closed disk of radius ε centered at z.
"ball"

$$= \{ w \in \mathbb{C} : |z - w| \leq \varepsilon \}.$$

- A general set S in $\mathbb{C}$ is called <u>open</u> if $\forall \omega \in S$, $\exists \varepsilon > 0$ s.t. $B(\omega, \varepsilon) \subseteq S$.

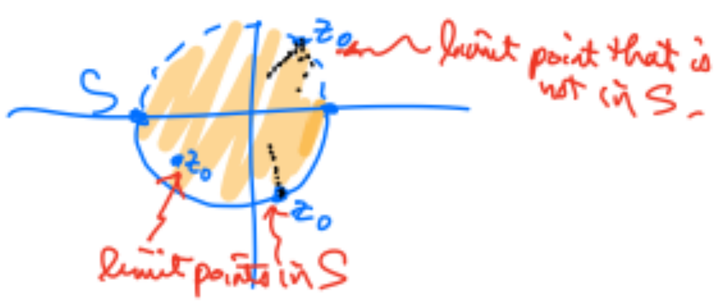
↑ is a subset of / is contained in.

Intuitively: does not contain its boundary, and it is 2-d.

- A general set F in $\mathbb{C}$ is called <u>closed</u> if F^c is open $\left(F^c = \{ z \in \mathbb{C} : z \notin F \} = \text{complement of } F. \right)$.

<u>Intuitively</u>: contains its limit points (including its boundary).

- A <u>limit point</u> of a set S is a point z_0 of $\mathbb{C}$ [could be in S or in S^c] such that $\forall \varepsilon > 0$, $B(z_0, \varepsilon) \cap S \neq \emptyset$ and $B(z_0, \varepsilon) \cap S \neq \{z_0\}$. (ie $B(z_0, \varepsilon) \cap S \setminus \{z_0\} \neq \emptyset$).



Alternate Defn of closed set:

A set $S \subseteq \mathbb{C}$ is <u>closed</u> if it contains all of its limit pts.